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On a universal chain problem

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Abstract

M. S. Paterson introduced the idea of a universal chain for a set of points in a compact metric space. We consider the universal chain problem in a finite discrete space, and give a precise characterisation of its solution. Two applications are discussed.

1. Introduction

Consider the following simple parallel memory allocation problem. There are k processors, running independent programs, which share n equal sized blocks of memory. The memory blocks are indistinguishable apart from their address. If each processor is connected to each block, then the "connexion complexity" in the arrangement might be measured as kn. The situation with $k = 3$, $n = 9$ is diagrammed in Fig. 1. Suppose, however, we form *unequal* sized blocks of the same total capacity. It may be possible that we can still simulate any partition of memory in the original configuration, while having fewer blocks. A possible arrangement corresponding to Fig. 1 is shown in Fig. 2. There are now only five blocks, as against the original nine, but the reader may check that no possible partitions are excluded. We will show that it is generally true that relatively few blocks suffice, and will give a fairly precise characterisation of the minimum. Thus, if say $n \gg k \gg 1$, the connexion complexity can be reduced to a minimum of about $k^2 \log n$, while still allowing the available memory to be divided between the processors in any possible way. (Natural logarithms are used throughout unless the base is specified.)

The work here is inspired by that of Paterson [1] on "universal chains" in metric spaces. We will define a simple (possibly the simplest) universal chain problem, whose solution can be exactly determined. The problem is essentially that given above, but presented in a different setting. We will also give an application to proving lower bounds in more general universal chain problems.

The plan of the paper is as follows. In Section 2 we define universal chain problems, and outline Paterson's [1] results. In Section 3 we define our universal chain problem

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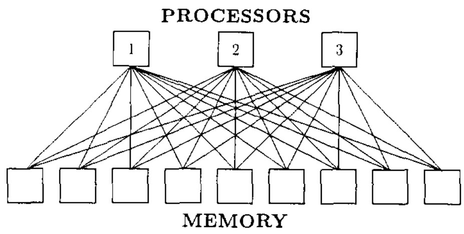

Fig. 1. Equal blocks ($k = 3$, $n = 9$).

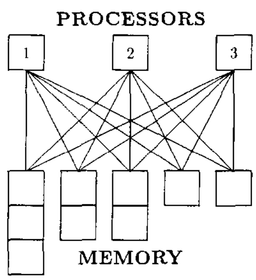

Fig. 2. Unequal blocks ($k = 3$, $n = 9$).

and present its solution. In Section 4 we discuss the applications of the result. Finally Section 5 gives some brief concluding remarks.

2. Universal chains

Paterson [1] introduced and investigated the notion of a *universal chain* for any multiset of n points in a fixed compact metric space $\mathcal{M}$, and gave applications to some wiring problems in chip design. Following [1], we make the following definition. An n-point *chain* C_n is an $(n-1)$-sequence $\langle a_1, a_2, \ldots, a_{n-1}\rangle$ of real numbers. The positions in the sequence are the *links*, and the a_i are their *lengths*. The length of C_n is $l_n = \sum_{i=1}^{n-1} a_i$. Let S_n be a multiset of cardinality n in a (compact) metric space $\mathcal{M} = (X, d)$. We say C_n *covers* S_n if there is an ordering $\langle x_1, x_2, \ldots, x_n\rangle$ of S_n such that $d(x_i, x_{i+1}) \leq a_i$ for $i = 1, 2, \ldots, n-1$. We say that C_n is *universal* if it covers *any* S_n in $\mathcal{M}$. The *universal chain problem* for $\mathcal{M}$ is to determine the minimum length $l_n^{\min}$ for an n-point universal chain.

Intuitively, we may think of the chain as a cord with n marks on it, of which two are the ends of the cord. We have to lay down the cord (slackly if necessary) on n given points so that one mark falls on each point. The chain (cord) is universal if we can do this wherever the points are placed. We wish to know the shortest cord which meets this objective.

Paterson [1] gave the following estimates for universal chain lengths. (In fact [1] is mainly concerned with order-of-magnitude growth of $l_n^{\min}$, but we give the precise bounds here for later reference.) In all cases, the metric is L_∞ distance, so we specify only the ground set for $\mathcal{M}$.

(1) For the unit interval [0, 1],

$$\tfrac{1}{12}(\log_2 n)^2 \le l_n^{\min} \le (\log 2)(\log_2 n)^3 + 1.$$

(The upper bound here is due to Leighton.)

(2) For the unit square $[0, 1]^2$,

$$\sqrt{n} + 1 \le l_n^{\min} \le 10\sqrt{n} - 1.$$

(3) For the unit hypercube $[0, 1]^d$,

$$1 \le \frac{l_n^{\min}}{n^{(d-1)/d}} \le c(d),$$

where $c(d) \to 1$ as $d \to \infty$.

The upper bounds are proved by exhibiting a chain construction and arguing its universality. The lower bounds are proved by examining certain "regularly spaced" sets in $\mathcal{M}$. Better lower bounds, though only by a constant factor, will be given in Section 4 below.

3. Finite discrete spaces

The problem we will consider here is that in which $\mathcal{M}$ is a finite discrete metric space. Thus we shall let $\mathcal{M}$ be $\mathcal{D}_k$, where $\mathcal{D}_k = (Y_k, d)$, $Y_k = \{y_1, y_2, \ldots, y_k\}$, $d(y_j, y_j) = 0$ and $d(y_i, y_j) = 1$ $(i \ne j)$. A multiset of points in $\mathcal{D}_k$ is simply a multiset of the y's.

Any n-point chain C_n in $\mathcal{D}_k$ is a sequence of $(n-1)$ 0's and 1's. For example, if

$$S = \{y_1, y_1, y_1, y_2, y_2, y_3, y_3\},$$

then we could have

$$\langle x_1, \ldots, x_7\rangle = \langle y_1, y_1, y_1, y_2, y_2, y_3, y_3\rangle$$

and $C_7 = \langle 0, 0, 1, 0, 1, 0\rangle$, or we could have

$$\langle x_1, \ldots, x_7\rangle = \langle y_1, y_2, y_3, y_1, y_2, y_3, y_1\rangle$$

and $C_7 = \langle 1, 1, 1, 1, 1, 1\rangle$. It is obvious that only the *number* and *length* of the sequences of 0's bounded by 1's really matters. Their ordering is unimportant since the "price" for changing from any y_j to any other is the same. Thus, if we wish to cover a particular y_j, it does not matter whether we do it next or later. For homogeneity, in order that all 0-sequences start with an initial 1, we will add a "dummy" 1 to the start of the chain. We denote this augmented chain by C'_n. Now let $c_1 \geq c_2 \geq \cdots \geq c_p$ be the sorted lengths of the 0-sequences (including their initial 1) in the chain. Thus p is the length of the chain (which is one more than the number of links in the chain). For example, if $C_7 = \langle 0, 1, 1, 0, 0, 0 \rangle$, then $C'_7 = \langle 1, 0, 1, 1, 0, 0, 0 \rangle$, which is equivalent to $\langle 1, 0, 0, 0, 1, 0, 1 \rangle$, so that $p = 3$, $c_1 = 4$, $c_2 = 2$ and $c_3 = 1$.

Clearly we have $c_i \geq 1$ $(i = 1, 2, \ldots, p)$ and $\sum_{i=1}^{p} c_i = n$. We can think of the c_i as *counters* of given value which can cover that number of *points* on any y_j (i.e., copies of y_j in the multiset).

We now give a simple and effective criterion for universality of a chain in $\mathscr{D}_k$.

Theorem 1. *Positive integers $c_1 \geq c_2 \geq \cdots \geq c_p$ determine an n-point universal chain in $\mathscr{D}_k$ if and only if*

$$\sum_{j=1}^{i-1} c_j + kc_i \leq n + k - 1 \quad (i = 1, 2, \ldots, p), \tag{1}$$

$$\sum_{j=1}^{p} c_j = n. \tag{2}$$

Proof. First we prove sufficiency by induction. Since (1) holds,

$$c_i \leq \left\lfloor \frac{n + k - 1 - \sum_{j=1}^{i-1} c_j}{k} \right\rfloor = \left\lceil \frac{n - \sum_{j=1}^{i-1} c_j}{k} \right\rceil .$$

When $c_1, c_2, \ldots, c_{i-1}$ have been placed, there must be some y_j with at least $\lceil (n - \sum_{j=1}^{i-1} c_j)/k \rceil$ uncovered points. Thus c_i can be placed. Since (2) holds, all points must eventually be covered.

Next we prove necessity. The necessity of (2) is obvious. Now, by considering the most uniform configuration (in which all y_j have either $\lfloor n/k \rfloor$ or $\lceil n/k \rceil$ points) it is clear that we must have $c_1 \leq \lceil n/k \rceil$. Thus (1) must hold for $i = 1$. We now argue by induction. Suppose $i \geq 2$ is the least such that (1) does not hold. Consider the particular multiset of size n which comprises $(n - (k-1)(c_i - 1))$ points on y_1 and $c_i - 1$ points on y_j for $j = 2, 3, \ldots, k$. Note that $(n - (k-1)(c_i - 1)) > 0$, since otherwise

$$n + k - 1 \leq (k-1)c_i \leq (k-1)c_{i-1} < kc_{i-1},$$

contradicting the choice of i. Thus, since $c_i \geq 1$, the configuration is well defined. Now, since $c_l \geq c_i$ for $l \leq i$, these must all be placed on y_1. However, $\sum_{l=1}^{i} c_l > n - (k-1)(c_i - 1)$ (the number of points on y_1), by the failure of (1) for i. This contradicts the universality of the sequence. $\square$

We now show how shortest universal chains can be determined. Let

$$m_i = \left\lceil \frac{n - \sum_{l=1}^{i-1} m_l}{k} \right\rceil \quad (i = 1, 2, \ldots). \tag{3}$$

Clearly m_i can be calculated recursively. The m_i are obviously nonincreasing, and $m_i \geq 1$ as long a $\sum_{l=1}^{i-1} m_l < n$. We also have $m_i \leq n - \sum_{l=1}^{i-1} m_l$ (since $\lceil x/k \rceil \leq x$ if x is a nonnegative integer). Thus define $p_k(n)$ to be the unique p such that $m_p > 0$ and $\sum_{l=1}^{p} m_l = n$. The m_i $(i = 1, 2, \ldots, p)$ obviously form a universal sequence by Theorem 1. We now show that $p_k(n)$ is the minimal length of a universal sequence. The proof follows easily from the following

Lemma 2. *For any universal sequence* $c_1 \geq c_2 \geq \cdots \geq c_i \geq \cdots$

$$\sum_{l=1}^{i} c_l \leq \sum_{l=1}^{i} m_l.$$

Proof. From Theorem 1, $c_1 \leq \lceil n/k \rceil = m_1$. We now use induction.

$$\begin{aligned}
\sum_{l=1}^{i} c_l &= \sum_{l=1}^{i-1} c_l + c_i \\
&\leq \sum_{l=1}^{i-1} c_l + \left(n + k - 1 - \sum_{l=1}^{i-1} c_l\right)\Big/ k, \quad \text{by Theorem 1} \\
&= (1 - 1/k) \sum_{l=1}^{i-1} c_l + (n + k - 1)/k \\
&\leq (1 - 1/k) \sum_{l=1}^{i-1} m_l + (n + k - 1)/k, \quad \text{by induction} \\
&= \sum_{l=1}^{i-1} m_l + \left(n + k - 1 - \sum_{l=1}^{i-1} m_l\right)\Big/ k.
\end{aligned}$$

So

$$\begin{aligned}
\sum_{l=1}^{i} c_l &\leq \sum_{l=1}^{i-1} m_l + \left\lfloor \left(n + k - 1 - \sum_{l=1}^{i-1} m_l\right)\Big/ k \right\rfloor, \quad \text{since the } c_l \text{ are integers,} \\
&= \sum_{l=1}^{i-1} m_l + \left\lceil \left(n - \sum_{l=1}^{i-1} m_l\right)\Big/ k \right\rceil, \\
&= \sum_{l=1}^{i-1} m_l + m_i \\
&= \sum_{l=1}^{i} m_l. \qquad \square
\end{aligned}$$

Theorem 3. *The value $p_k(n)$ gives the length of any shortest universal chain in $\mathscr{D}_k$.*

Proof. Suppose $c_1, c_2, \ldots, c_r$ is universal with $r < p$. Then from Theorem 1 and Lemma 2 we have

$$n = \sum_{l=1}^{r} c_l \leq \sum_{l=1}^{r} m_l < n,$$

which is an obvious contradiction. □

Corollary 4. *$\langle m_1, m_2, \ldots, m_p \rangle$ determines a shortest universal chain.*

Corollary 5. *For all $k \geq 1$, define $p_k(0) = 0$. Then, for all $n, k \geq 1$,*

$$\begin{aligned} p_k(n) &= 1 + p_k(n - \lceil n/k \rceil) \\ &= 1 + p_k(\lfloor (1 - 1/k)n \rfloor) \\ &= 1 + p_k(\lceil (1 - 1/k)(n - 1) \rceil). \end{aligned}$$

Proof. Consider the first step in the calculation of the m_i using (3). □

Let us make a few simple observations.

Lemma 6. *If $1 \leq n \leq k$, then $p_k(n) = n$, and if $k < n \leq 2k$, then $p_k(n) = k + \lfloor (n - k)/2 \rfloor$.*

Proof. Clearly $p_k(n) = n$ for all $n \leq k$. If $n > k$, $\lceil n/k \rceil = 2$, and the result follows by induction from Corollary 5. □

We now give explicit bounds on $p_k(n)$. To this end let us define

$$b_k(n) = \frac{\log((n + k - 1)/2k)}{\log(k/(k - 1))} + k, \qquad u_k(n) = \frac{\log(n/k)}{\log(k/(k - 1))} + k,$$

for $k > 1$, and $b_1(n) = u_1(n) = 0$ for all $n > 0$. It is obvious that both $b_k(n)$ and $u_k(n)$ are increasing with n. It is also easily verified that

$$b_k(n) = 1 + b_k((1 - 1/k)(n - 1)), \quad u_k(n) = 1 + u_k((1 - 1/k)n). \tag{4}$$

Theorem 7. *For all $n \geq k \geq 1$, $\lceil b_k(n) \rceil \leq p_k(n) \leq \lfloor u_k(n) \rfloor$.*

Proof. By induction on n, using Corollary 5. First we establish, as basis, that the bounds are correct for $k \leq n \leq 2k$. This is easily checked for $n = k$. For $k < n \leq 2k$, we use Lemma 6.

For the lower bound, setting $x = (n - k - 1)/2$, we need to show (after some rearrangement) that the function

$$f(x) = \log(1 + x/k) - x\log(k/(k-1))$$

is never positive for $0 \leq x \leq (k-1)/2$. This can be proved by noting $f(0) = 0$, and showing $f'(x) \leq 1/k + \log(1 - 1/k) < 0$ for $x \geq 0$. (Thus in fact $f(x) < 0$ for all $x > 0$.)

For the upper bound, setting $x = n - k$, we must similarly show that, for $0 < x \leq k$, the function

$$g(x) = 2\log(1 - x/k) - x\log(k/(k-1))$$

is never negative. Again $g(0) = 0$, and since g is clearly concave, the result follows if $g(k) = 2\log 2 - k\log(k/(k-1)) \geq 0$. This is a consequence of the fact that $k\log(k/(k-1))$ decreases with k, which is easily established by expanding the logarithm in powers of $1/k$.

We now move to the main induction. We have $n \geq 2k + 1$, and so

$$(1 - 1/k)n > (1 - 1/k)(n-1) \geq (1 - 1/k)(2k) \geq k,$$

since $k \geq 2$. Now, by Corollary 5 and (4),

$$\begin{aligned} p_k(n) &= 1 + p_k(\lfloor (1 - 1/k)n \rfloor) \\ &\leq 1 + u_k((1 - 1/k)n) \\ &= u_k(n), \end{aligned}$$

and

$$\begin{aligned} p_k(n) &= 1 + p_k(\lceil (1 - 1/k)(n-1) \rceil) \\ &\geq 1 + b_k((1 - 1/k)(n-1)) \\ &= b_k(n), \end{aligned}$$

completing the induction. □

Corollary 8. $p_2(n) = \lceil \log_2(n+1) \rceil$.

Proof. Putting $k = 2$ in Theorem 7 gives

$$\lceil \log_2(n+1) \rceil \leq p_2(n) \leq 1 + \lfloor \log_2 n \rfloor.$$

But it is easy to show that the two bounds in this expression are equal. □

It is instructive to compare the result of Theorem 7 with the lower bound argument of Paterson [1], which for our problem is as follows. Since there are $\binom{n+k-1}{k-1}$ possible configurations of points, and a chain with p 1-links can make at most k^p choices of layout, it follows we must have

$$p_k(n) \geq \frac{\log\binom{n+k-1}{k-1}}{\log k} = b'_k(n),$$

say. For $k = 2$, this gives the correct bound, but for $k > 2$, it is too small. As $n \to \infty$, $b_k(n) \sim u_k(n) \sim \log n/\log(k/(k-1))$. However, $b'_k(n) \sim (k-1)\log n/\log k$, which is too small by a factor which decreases (to zero) as k increases.

4. Applications

The solution of the memory management problem of Section 1 should now be clear. We simply choose blocks whose sizes are the c_i for the n-point universal chain in $\mathscr{D}_k$.

We may also apply the results to proving lower bounds for universal chains in general compact metric spaces. We will need the following

Lemma 9. $b_{k+1}(n) - b_k(n) > \log((n+k-1)/(2k))$ *for* $k \geq 1$.

Proof.

$$\begin{aligned} b_{k+1}(n) - b_k(n) &= 1 + \frac{\log((n+k)/2(k+1))}{\log((k+1)/k)} - \frac{\log((n+k-1)/2k)}{\log(k/(k-1))} \\ &= \frac{\log((n+k)/2k)}{\log((k+1)/k)} - \frac{\log((n+k-1)/2k)}{\log(k/(k-1))} \\ &\geq \log((n+k-1)/2k)\left(\frac{1}{\log((k+1)/k)} - \frac{1}{\log(k/(k-1))}\right) \\ &\geq \log((n+k-1)/2k), \end{aligned}$$

since

$$\frac{1}{\log((k+1)/k)} - \frac{1}{\log(k/(k-1))} > 1 \quad \text{for all } k > 1. \tag{5}$$

Equation (5) may be proved as follows. Letting $x = 1/k$, it amounts to showing that the function

$$F(x) = \log(1+x)\log(1-x) - \log(1+x) - \log(1-x)$$

is positive for $0 < x < 1$. Now $F(0) = 0$, and $F'(x) = G(x)/(1-x^2)$, where

$$G(x) = 2x - (1+x)\log(1+x) + (1-x)\log(1-x).$$

So we need only show that $G(x) > 0$ for $0 < x < 1$. However $G(0) = 0$, and $G'(x) = -\log(1-x^2) > 0$, so (5) follows. $\square$

In fact, the difference between the two sides in (5) is always less than 0.024 (its value for $k = 2$). Thus the bound of Lemma 9 is reasonably tight.

Now suppose $\mathcal{M} = (X, d)$ is any compact metric space. By a *ball* of radius r and centre x in $\mathcal{M}$ we will mean

$$\{y \in X : d(x, y) < r\}.$$

(This is an "open ball" in the usual parlance.) For $k = 1, 2, \ldots$ let ρ_k be the largest number ρ such that the centres of k nonintersecting balls of radius $\frac{1}{2}\rho$ can be placed in $\mathcal{M}$. Clearly $\rho_1 = \infty$, and ρ_k is nonincreasing with k. Now, if we place n points in $\mathcal{M}$ using only the k ball centres, the distance between points is either 0 or at least ρ_k. Therefore, recalling that $p_k(n)$ is one more than the number of links in the universal chain of Section 3, the number of links of length at least ρ_k in a universal chain C_n must be at least $p_k(n) - 1$. Let λ_k be the number of links i such that $\rho_k \le a_i < \rho_{k-1}$ for $k = 2, 3, \ldots$. Thus,

$$\sum_{i=2}^{k} \lambda_i \ge p_k(n) - 1 \quad (k = 2, 3, \ldots, n). \tag{6}$$

The length of C_n clearly satisfies

$$l_n \ge L = \sum_{k=2}^{n} \rho_k \lambda_k.$$

Thus $l_n^{\min}$ is at least the minimum of L subject to (6), $L_{\min}$ say. Multiply the kth equation in (6) by $(\rho_k - \rho_{k+1})$ for $k = 2, 3, \ldots, n-1$, and the nth by ρ_n. After rearrangement of the left-hand side, this gives

$$L \ge \rho_n(p_n(n) - 1) + \sum_{k=2}^{n-1} (\rho_k - \rho_{k+1})(p_k(n) - 1) \tag{7}$$

$$= \sum_{k=2}^{n} \rho_k(p_k(n) - p_{k-1}(n)). \tag{8}$$

However, putting $\lambda_k = p_k(n) - p_{k-1}(n)$, gives

$$L_{\min} \le \sum_{k=2}^{n} \rho_k(p_k(n) - p_{k-1}(n)),$$

and thus $L_{\min}$ is given by the right-hand side of either (7) or (8). Also, comparing (7) with (8), it is clear that replacing $p_k(n)$ with $b_k(n)$ will give a lower bound on $L_{\min}$. Thus

$$l_n^{\min} \ge \sum_{k=2}^{n} \rho_k(b_k(n) - b_{k-1}(n))$$

$$> \sum_{k=1}^{n-1} \rho_{k+1} \log((n + k - 1)/2k), \tag{9}$$

using Lemma 9.

We will apply (9) to the unit interval and square, in each case using the induced L_∞ metric, as in [1]. We are, of course, looking particularly at the case of large n.

First, in $[0, 1]$, clearly $\rho_k = 1/(k-1)$ for $k = 2, 3, \ldots$, using ball centres equally spaced at $j/(k-1)$ $(j = 0, 1, 2, \ldots, k-1)$. Thus

$$l_n^{\min} > \sum_{k=1}^{n-1} \frac{\log((n+k-1)/2k)}{k}. \tag{10}$$

Since the summand clearly decreases with k and is 0 when $k = n-1$, we have

$$l_n^{\min} > \int_1^{n-1} \frac{\log((n+x-1)/2x)}{x} \, \mathrm{d}x = I_n,$$

say. Now substituting $y = (n-1)/x$ gives

$$\begin{aligned} I_n &= \int_1^{n-1} \frac{\log((1+y)/2)}{y} \, \mathrm{d}y \\ &> \int_1^{n-1} \frac{\log((1+y)/2)}{1+y} \, \mathrm{d}y \\ &= \left[\tfrac{1}{2}(\log((1+y)/2))^2\right]_1^{n-1} \\ &= \tfrac{1}{2}\log^2(n/2). \end{aligned}$$

So $l_n^{\min} \geq \frac{1}{2}\log^2(n/2)$. It is not difficult to show that the right-hand side in (10) is $\frac{1}{2}\log^2 n \pm \mathrm{O}(\log n)$, so tighter estimations produce no asymptotic improvement. Note that we obtain almost a threefold improvement of the lower bound in [1] (see Section 2), of $(\log_2 n)^2/12 \approx 0.17 \log^2 n$.

Next we consider $[0, 1]^2$. Let $l = \lfloor \sqrt{k} \rfloor$. Consider the regular grid of ball centres $(i/l, j/l)$ $(i, j = 0, 1, 2, \ldots, l)$. This contains $(l+1)^2 > k$ balls, each of radius $\frac{1}{2}(1/l) \geq \frac{1}{2}(1/\sqrt{k})$. Thus we have $\rho_k \geq 1/\sqrt{k}$. Therefore from (9) we have

$$\begin{aligned} l_n^{\min} &> \sum_{k=1}^{n-1} \frac{\log((n+k-1)/2k)}{\sqrt{k+1}} \\ &> \int_1^{n-1} \frac{\log((n+x-1)/2x)}{\sqrt{x+1}} \, \mathrm{d}x \\ &= 4\sqrt{n-2}\,(\arctan\sqrt{n/(n-2)} - \arctan\sqrt{2/(n-2)}) \\ &\quad - 2\sqrt{2}\log(n/2) + 2\log(n-1) \\ &\quad - 4\log(\sqrt{n}+1) + 4\log(\sqrt{2}+1), \end{aligned} \tag{11}$$

by elementary methods of integration. For large n the right-hand side of (11) is $\pi\sqrt{n} - \mathrm{O}(\log n)$. Asymptotically, this gives more than a threefold improvement of the lower bound in [1].

5. Concluding remarks

It is clear that the universal chain problem discussed here is only tractable because of its simplicity. In general, universal chain problems seem difficult to solve exactly, and there are obvious complexity questions which arise. For example, we have given a polynomial time criterion for deciding the universality of a chain in our problem (Theorem 1). In general, it is not clear that (a properly specified version of) this problem will even be in NP $\cup$ co-NP (though it is clearly no higher than the second level of the polynomial hierarchy). Thus, in general, there seems little hope of doing anything more than roughly estimating the minimum lengths of universal chains, as in [1]. We have slightly improved the lower bounds of [1], but it appears that not even these lower bound techniques capture much of the structure of the general problem. However, the problem we have solved does seem to have some intrinsic interest. As a final illustration, the reader might use a small extension of the methods of this paper to solve the following puzzle, due to Paterson [2].

An overseer sets out to pay the seven labourers who have tilled (all or part of) a hundred acre field. The rate of pay is one link of gold chain for each complete acre tilled. Having no advance knowledge of how many acres each labourer has completed, what is the minimum number of pieces of gold chain the overseer must carry? (He cannot break a chain after leaving.)

Acknowledgement

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